

STABILITY OF PID-CONTROLLED LINEAR TIME-DELAY FEEDBACK SYSTEMS

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ABSTRACT. The stability of feedback systems consisting of linear time-delay plants and PID controllers has been investigated for many years by means of several methods, of which the Nyquist criterion, a generalization of the Hermite-Biehler Theorem, and the root location method are well known. The main purpose of these researches is to determine the range of controller parameters that allow stability. Explicit and complete expressions of the boundaries of these regions and computation procedures with a finite number of steps are now available only for first-order plants, provided with one time delay. In this note, the same results, based on Pontryagin's studies, are presented for arbitrary-order plants.

1. INTRODUCTION

The feedback structure considered in this note is depicted in Fig. 1 and the related transfer functions of the process $P(s)$ and the controller $C(s)$ are given by

$$(1.1) \quad P(s) = K \frac{P_n(s)}{P_d(s)} e^{-Ls} = K \frac{\prod_{i=1}^{i=m} (1 + Z_i s)}{\prod_{i=1}^{i=n} (1 + T_i s)} e^{-Ls}$$

$$(1.2) \quad C(s) = K_p + \frac{K_i}{s} + K_d s,$$

where K is the plant steady-state gain, T_i and Z_i the plant time constants, L is the positive plant time delay and K_p , K_i and K_d are the parameters of the PID controller.

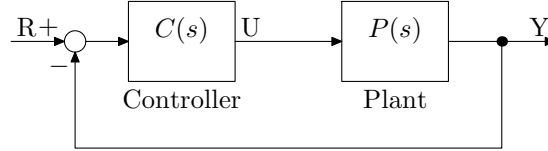


FIGURE 1. Feedback control system

Complete explicit expressions of the boundaries of the stability regions in first-order plants have been found in [1] with a version of the Hermite-Biehler Theorem derived by Pontryagin, in [2] with the Nyquist criterion, and in [3] with the root location method. Moreover the second-order plants have been investigated in [4] by

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means of a graphical approach; the results obtained are correct, but the stability conditions are not all explicit and no finite number of required computation steps is specified. Finally arbitrary-order plants have been studied with the Nyquist criterion in [5], but P and PID controllers with a given K_p separately are considered and no information about the set of the process parameters that allow stability is given.

This note can be considered as an extension of [1] to the arbitrary-order plants and is organized as follows. In Section 2 all the analytical expressions that will be used in the next sections are evaluated in detail. In Section 3 a process transfer function without zeros is considered and the stability regions are explicitly evaluated by means of a version of the Hermite-Biehler Theorem derived by Pontryagin, already used in [1]. A second-order plant is studied as example and the related stability regions are determined and plotted in two figures. In Section 4 a process transfer function with zeros is considered and the stability regions are found by means of a new theorem. The two procedures of the Sections 3 and 4 are essentially equal, consist of a finite number of steps and yield the stability regions in both process and controller parameters planes. In Section 5 some conclusive remarks are given.

The importance of explicit expressions of the boundaries of the stability zones has been enhanced by the introduction of the controller tuning charts in [6] (used also in [7]).

2. PRELIMINARIES

The closed-loop transfer function $T(s)$ of the system is given by

$$(2.1) \quad T(s) = \frac{K(K_i + K_p s + K_d s^2)P_n(s)}{sP_d(s)e^{Ls} + K(K_i + K_p s + K_d s^2)P_n(s)}.$$

According to the Pontryagin's studies, presented in [8] and summarized in [9], it is necessary that $T(s)$ has a bounded number of poles with arbitrary large positive real part for stability. This holds if the denominator of $T(s)$ has a principal term $a_p q s^p e^{q s}$ (in our case, where $p = n + 1$ and $q = 1$, it exists if $m \leq n - 1$) and the function $\chi_p(s)$, coefficient of s^p , (in our case $\chi_p(s) = e^{q s} \prod_{i=1}^{i=n} T_i$ for $m < n - 1$ and $\chi_p(s) = e^{q s} \prod_{i=1}^{i=n} T_i + K K_d \prod_{i=1}^{i=m} Z_i$ for $m = n - 1$) has all the zeros in the open left half plane. This happens if one of the following conditions is satisfied:

(a) $m < n - 1$

(b) $m = n - 1$ and $\left| K K_d \prod_{i=1}^{i=m} Z_i \right| < \left| \prod_{i=1}^{i=n} T_i \right|.$

The denominator of $T(s)$, given by (2.1), divided by $P_n(s)/L$ and hence named $H(s)$, can be written, according to (1.1), as

$$(2.2) \quad H(s) = L s e^{Ls} \frac{\prod_{i=1}^{i=n} (1 + T_i s)}{\prod_{i=1}^{i=m} (1 + Z_i s)} + L K (K_i + K_p s + K_d s^2).$$

Since all the poles of the closed-loop transfer function $T(s)$ are zeros of $H(s)$ and a system is stable if no pole of $T(s)$ lies in the right half-plane, the above system is stable if no zero of $H(s)$ lies in the right half-plane. For process transfer functions without zeros, examined in Section 3, $H(s)$ is a quasi-polynomial and a version of the Hermite-Biehler Theorem derived by Pontryagin is employed. For process transfer functions with zeros, examined in Section 4, $H(s)$ is a quasi-polynomial

divided by a polynomial and a new theorem, proved by use of the Principle of the Argument, is employed.

Now, before the explanation of the proposed procedures, let us evaluate all the expressions that will be used in the next sections. It is convenient to introduce the normalized time referred to the plant time delay L and the dimensionless parameters $\sigma = L s$, $t_i = T_i/L$, $z_i = Z_i/L$, $h = K K_p$, $h_i = K K_i L$ and $h_d = K K_d/L$, in order to obtain equations independent of the real values of the parameters. Applying these simplifications, (2.2) becomes

$$(2.3) \quad H(\sigma) = \sigma e^{\sigma} \frac{\prod_{i=1}^{i=n} (1 + t_i \sigma)}{\prod_{i=1}^{i=m} (1 + z_i \sigma)} + h_i + h \sigma + h_d \sigma^2.$$

Moreover, assuming $P_d(j y/L) = A(y) + jB(y)$ and $P_n(j y/L) = C(y) + jD(y)$, the real and the imaginary components $F(y)$ and $G(y)$ of $H(\sigma)$, calculated for $\sigma = j y$, are given by

$$(2.4) \quad F(y) = h_e - F_1(y)$$

$$(2.5) \quad G(y) = y[h - G_1(y)]$$

where

$$(2.6) \quad F_1(y) = y[Q(y)\cos(y) + P(y)\sin(y)]$$

$$(2.7) \quad G_1(y) = -P(y)\cos(y) + Q(y)\sin(y)$$

$$(2.8) \quad \begin{aligned} h_e &= h_i - h_d y^2 \\ P(y) &= \frac{A(y)C(y) + B(y)D(y)}{C^2(y) + D^2(y)} \\ Q(y) &= \frac{-A(y)D(y) + B(y)C(y)}{C^2(y) + D^2(y)} \end{aligned}$$

$$(2.9) \quad A(y) = 1 + \sum_{i=1}^{i=\text{int}(n/2)} U(n, 2i) (-1)^i y^{2i}$$

$$(2.10) \quad B(y) = \sum_{i=0}^{i=\text{int}((n-1)/2)} U(n, 2i+1) (-1)^i y^{2i+1}$$

$$(2.11) \quad C(y) = 1 + \sum_{i=1}^{i=\text{int}(m/2)} V(m, 2i) (-1)^i y^{2i}$$

$$(2.12) \quad D(y) = \sum_{i=0}^{i=\text{int}((m-1)/2)} V(m, 2i+1) (-1)^i y^{2i+1}.$$

For sake of clarity, U and V are the symmetric expressions of the time constants t_i and z_i ; $U(n, k)$ is the sum of the $\binom{n}{k}$ products of k different t_i selected among the total n (for example $U(3, 2) = t_1 t_2 + t_2 t_3 + t_3 t_1$ and $U(3, 3) = t_1 t_2 t_3$).

The derivative of $G(y)$ with respect to y is given by

$$(2.13) \quad G'(y) = h - G_1(y) - y G'_1(y).$$

Assuming $G(y) = 0$ and $h = -1$ in (2.5), one obtains $\tan(y/2) = E(y)$ where

$$(2.14) \quad E(y) = \frac{-Q(y) \pm \sqrt{P^2(y) + Q^2(y) - 1}}{1 + P(y)}.$$

It is easy to check that $E(y)$ exists for $y = 0$ only if $\sum_{i=1}^{i=n} t_i^2 > \sum_{i=1}^{i=m} z_i^2$. The derivative of $E(y)$ with respect to y , evaluated at $y = 0$ and named $E'(0)$, is given by

$$(2.15) \quad \begin{aligned} E'(0) &= +0.5(-U(n, 1) + V(m, 1)) \\ &\pm 0.5\sqrt{U^2(n, 1) - 2U(n, 2) - V^2(m, 1) + 2V(m, 2)}. \end{aligned}$$

Denoting by $E_-(y)$ and $E_+(y)$ the two branches of $E(y)$ related respectively to the minus and plus signs, their derivatives $E'_-(0)$ and $E'_+(0)$ are higher than the derivative of $\tan(y/2)$, equal to 0.5, depending on Φ_1 and Φ_2 , given by

$$(2.16) \quad \Phi_1 = 1 + U(n, 1) - V(m, 1)$$

$$(2.17) \quad \begin{aligned} \Phi_2 &= +1 + 2U(n, 1) + 2U(n, 2) - 2U(n, 1)V(m, 1) \\ &+ 2V^2(m, 1) - 2V(m, 1) - 2V(m, 2). \end{aligned}$$

In detail, the number of the derivatives $E'_-(0)$ and $E'_+(0)$ higher than 0.5 are the following: zero if $\Phi_1 > 0$ and $\Phi_2 > 0$, one if $\Phi_2 < 0$, and two if $\Phi_1 < 0$ and $\Phi_2 > 0$. Let us denote by E_d

$$(2.18) \quad E_d = \tan(y_t/2) - E(y_t)$$

where y_t , corresponding to equal derivatives with respect to y of $\tan(y/2)$ and $E(y)$, is a root of $0.5(1 + \tan^2(y_t/2)) = E'(y_t)$.

Differentiating (2.7) with respect to y once and twice, one obtains

$$(2.19) \quad \frac{dG_1(y)}{dy} = (-P'(y) + Q(y))\cos(y) + (P(y) + Q'(y))\sin(y)$$

$$(2.20) \quad \begin{aligned} \frac{d^2G_1(y)}{dy^2} &= +(+P(y) - P''(y) + 2Q'(y))\cos(y) \\ &+ (-Q(y) + Q''(y) + 2P'(y))\sin(y). \end{aligned}$$

It is worthwhile to note that $G_1(0) = -1$, $dG_1(0)/dy = 0$, and $d^2G_1(0)/dy^2 = \Phi_2$, where Φ_2 is given by (2.17). Evaluating $h_m = |G_1(y_m)|$, where y_m is a root of $dG_1(y)/dy$ given by (2.19), one obtains

$$(2.21) \quad h_m = \frac{|P^2(y_m) + Q^2(y_m) + P(y_m)Q'(y_m) - P'(y_m)Q(y_m)|}{\sqrt{(P'(y_m) - Q(y_m))^2 + (P(y_m) + Q'(y_m))^2}}.$$

Eliminating $\cos(y)$ and $\sin(y)$ from $F(y) = 0$ and $G(y) = 0$, given by (2.4) and (2.5), yields

$$(2.22) \quad h_e = y\sqrt{P^2(y) + Q^2(y) - h^2}.$$

Denote by Γ_{ai} and Γ_{bi} the two straight lines whose equations in the (h_i, h_d) -plane are obtained introducing in (2.8) respectively $y = y_{ai}$, $h = h_{eai}$ and $y = y_{bi}$, $h = h_{ebi}$ (see Figs. 2 and 5); denote further by V_i , U_i and W_i the vertices of a triangle, whose sides are the axis h_d and the two lines Γ_{ai} and Γ_{bi} . The coordinates of these vertices are given by

$$(2.23) \quad h_i(V_i) = \frac{y_{bi}^2 h_{eai} - y_{ai}^2 h_{ebi}}{y_{bi}^2 - y_{ai}^2}; \quad h_d(V_i) = \frac{-h_{ebi} + h_{eai}}{y_{bi}^2 - y_{ai}^2}$$

$$(2.24) \quad h_d(U_i) = -h_{ebi}/y_{bi}^2; \quad h_d(W_i) = -h_{eai}/y_{ai}^2.$$

Considering (2.22), the coordinates $h_d(R_i)$ and $h_d(S_i)$ of the points lying on Γ_{ai} and Γ_{bi} at $h_i = h_i(V_1)$ are given by

$$(2.25) \quad h_d(R_i) = \frac{h_i(V_1)}{y_{bi}^2} - \text{sign}[h_{ebi}] \frac{\sqrt{P^2(y_{bi}) + Q^2(y_{bi}) - h^2}}{y_{bi}}$$

$$(2.26) \quad h_d(S_i) = \frac{h_i(V_1)}{y_{ai}^2} - \text{sign}[h_{eai}] \frac{\sqrt{P^2(y_{ai}) + Q^2(y_{ai}) - h^2}}{y_{ai}}.$$

It is easy to check that, when $i = \infty$, the absolute values of $h_d(U_i)$ and $h_d(W_i)$, if $m < n - 1$, are equal to ∞ and, if $m = n - 1$, to $H_d(\infty)$ given by

$$(2.27) \quad H_d(\infty) = |U(n, n)/V(m, m)|.$$

3. PROCESS TRANSFER FUNCTION WITHOUT ZEROS

When the process transfer function does not have zeros, $m = 0$ and thus $C(y) = 1$, $D(y) = 0$, $P(y) = A(y)$, $Q(y) = B(y)$ hold; therefore, the function $H(\sigma)$, given by (2.3), is a quasi-polynomial and the Pontryagin's results are integrally applicable. The following two conditions derived from Theorem 3.2 of [1] and from Theorem 13.7 of [9], respectively, must be satisfied in order to have a stable system:

- Condition no. 1

Consider that the principal term of $H(\sigma)$, given by (2.3), is $\sigma^{n+1}e^\sigma$, set $H(jy) = F(y) + jG(y)$ and let ϵ be an appropriate constant such that the coefficient of y^{n+1} in $G(y)$ does not vanish at $y = \epsilon$. The number N_r of the real roots of $G(y)$ in the interval $-2r\pi + \epsilon \leq y \leq 2r\pi + \epsilon$ for sufficiently large r must be

$$(3.1) \quad N_r = 4r + n + 1.$$

- Condition no. 2

For all the zeros $y = y_0$ of the function $G(y)$ the inequality $G'(y_0)F(y_0) - G(y_0)F'(y_0) > 0$, that is $G'(y_0)F(y_0) > 0$, must hold.

In order to study both stable and unstable free-delay plants, the following two cases, adopted also in [1], are considered:

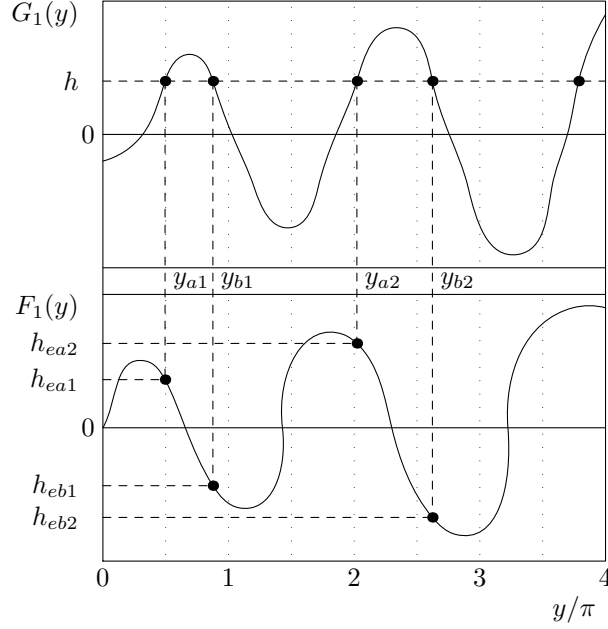
- (1) $U(n, n) > 0$ (even number of negative plant time constants T_i)
 $h > -1$ and $h_i > 0$.
- (2) $U(n, n) < 0$ (odd number of negative plant time constants T_i)
 $h < -1$ and $h_i < 0$.

From (2.5), (2.9) and (2.10) it follows that the coefficient of the highest degree of y in $G(y)$ is $U(n, n)\cos(y)$ when n is even and $U(n, n)\sin(y)$ when odd; hence we assume $\epsilon = 0$ if n is even and $\epsilon = 0.5\pi$ if odd.

Typical functions $F_1(y)$ and $G_1(y)$ are plotted in Fig. 2; according to (2.5) there is one root of $G(y)$ at $y = 0$ and one for each intersection of $G_1(y)$ with the horizontal line having the ordinate equal to a given h .

Denoting by N_e the number of the intersections between $G_1(y)$ and $h = -1$ corresponding to N_r in Fig. 2 and assuming that no local minimum or maximum of $G_1(y)$ is equal to -1 for $y \neq 0$, the relationship between N_e and N_r is given by

$$(3.2) \quad \begin{aligned} N_e &= N_r - 1 = 4r + n \quad \text{for } U(n, n)\Phi_2 < 0 \\ N_e &= N_r - 3 = 4r + n - 2 \quad \text{for } U(n, n)\Phi_2 > 0, \end{aligned}$$

FIGURE 2. Typical plots of $F_1(y)$ and $G_1(y)$

where Φ_2 is according to (2.17); (3.2) can be easily checked considering that from (2.7), (2.19) and (2.20) it follows $G_1(0) = -1$, $G_1'(0) = 0$ and $G_1''(0) = \Phi_2$, and also that $h > -1$ must hold if $U(n, n) > 0$ and $h < -1$ if $U(n, n) < 0$. Since $h = -1$ is the common limit value of h for the two considered cases ($U(n, n) < 0$ and $U(n, n) > 0$), the existence of the N_e intersections given by (3.2) represents a prerequisite of a plant to be made stable.

The number N_e can be evaluated by counting the intersections of the plots of $\tan(y/2)$ and $E(y)$, given by (2.14). From (2.14) it follows that $E(y)$ is an odd function of y ; moreover, assuming $S_a = \text{sign}[A(+\infty)]$ and $S_b = \text{sign}[B(+\infty)]$, $E_-(+\infty)$ and $E_+(+\infty)$ can be expressed as

- n even
 $E_-(+\infty) = -S_a 1$ and $E_+(+\infty) = +S_a 1$.
- n odd
 $E_-(+\infty) = -S_a \infty$ for $S_b > 0$ and $E_-(+\infty) = -S_a 0$ for $S_b < 0$,
 $E_+(+\infty) = +S_a 0$ for $S_b > 0$ and $E_+(+\infty) = +S_a \infty$ for $S_b < 0$.

If $E(y)$ has no pole, splitting N_e into N_{e1} ($|y| < \pi$) and N_{e2} and considering the above described behavior of $E(y)$ at $y = \infty$ and also at $y = 0$, one obtains

- $|y| < \pi$
 $N_{e1} = 0$ if $\Phi_1 > 0$ and $\Phi_2 > 0$,
 $N_{e1} = 2$ if $\Phi_2 < 0$,
 $N_{e1} = 4$ if $\Phi_1 < 0$ and $\Phi_2 > 0$,
 where Φ_1 and Φ_2 are given by (2.16) and (2.17).
- $-2r\pi + \epsilon \leq y \leq -\pi$; $\pi \leq y \leq 2r\pi + \epsilon$ (see Fig. 3 (a))
 $N_{e2} = 4r - 2$ if n is even,
 $N_{e2} = 4r - 1$ if n is odd and $A(+\infty)B(+\infty) > 0$,

$$N_{e2} = 4r - 3 \text{ if } n \text{ is odd and } A(+\infty)B(+\infty) < 0, \\ \text{where } \text{sign}[A(+\infty)B(+\infty)] = \text{sign}[-(-1)^n U(n, n-1)U(n, n)] .$$

It is clear that, if $E(y)$ has no pole, N_e will be always lower than the value required by (3.2) for enough large n . A positive solution can be reached only if $E(y)$ is provided with a suitable number of poles, since N_{e2} is increased by one for each added pole (see Fig. 3); this happens if $E_d > 0$ for case (b1), if $E_d < 0$ for case (b2) and without further condition for case (b3), where E_d is given by (2.18). Since the denominator of $E(y)$ is a polynomial of y^2 of degree $d = \text{int}(n/2)$, the maximum number of poles of $E(y)$ is equal to $2d$ and the actual number can be determined by means of the Sturm Theorem, as detailed in Appendix A.

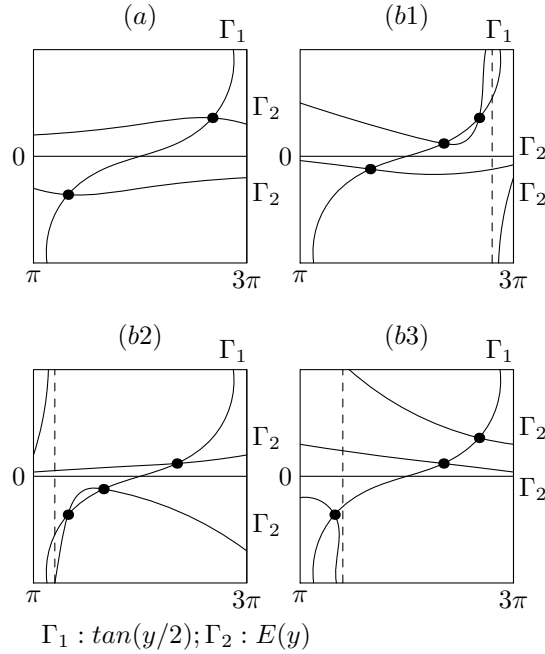


FIGURE 3. Plots of $\tan(y/2)$ and $E(y)$ for $y > \pi$

The sought-after procedure can be detailed as follows:

- (1) Process parameters (see Fig. 4)

The stability region in the process parameters plane is that where (3.2) holds; its boundary line is a proper set of the boundary lines of the zones with different numbers N_e , that is, $\Phi_1 = 0$ and $\Phi_2 = 0$ according to (2.16) and (2.17) and $\Psi_{i,j} = 0$, as explained in Appendix A, and eventually $E_d = 0$ given by (2.18). Since the expressions of these boundary lines are functions of $t_i = T_i/L$, it is possible to evaluate the stability range of L when the parameters T_i are known and, conversely, of each T_i when L and the remaining T_i are known. Moreover, if one needs only to know if a given plant can be made stable, it is not necessary to determine the stability regions, but it is enough to examine the plots of $E(y)$ and $\tan(y/2)$ and to check whether the number of the intersections N_e satisfies (3.2).

(2) Controller parameter h (see Fig. 2)

The requirement, stated as Condition no. 1, is fulfilled if the selected value of h is included in the interval from -1 to h_p for $U(n, n) > 0$ or from h_n to -1 for $U(n, n) < 0$, where h_p and h_n are respectively the local maxima and minima of $G_1(y)$ nearest to -1 ; both can be calculated by introducing in (2.7) the related root of $dG_1(y)/dy$ obtained from (2.19). A finite number of these maxima or minima must be examined in order to find the nearest to -1 , exactly up to the first value of y_m higher than y_{r1} . This limit y_{r1} is the largest positive root of the derivative with respect to y_m of h_m , since h_m , given by (2.21), monotonically increases for $y_m > y_{r1}$. It is obvious that, if such root does not exist, only the first value must be considered.

(3) Controller parameters h_i, h_d (see Fig. 5)

Considering (2.4), (2.8) and (2.13), the requirement, stated as Condition no. 2, is fulfilled if the following inequalities

$$(3.3) \quad \begin{aligned} h_i - h_d y_0^2 &< F_1(y_0) \quad \text{if} \quad G'_1(y_0) > 0 \\ h_i - h_d y_0^2 &> F_1(y_0) \quad \text{if} \quad G'_1(y_0) < 0 \end{aligned}$$

hold for each root $y = y_0$ of $G(y)$, given by (2.5), evaluated with a value of h included in the above specified interval.

The stability region in the (h_i, h_d) -plane consists of the intersection of a finite number of triangles; each of them is related to a couple of roots y_{ai} and y_{bi} of $G(y)$, has the axis h_d and the two straight lines given by (3.3) as sides and the points U_i, V_i and W_i as vertices, whose coordinates are given by (2.23) and (2.24) (see Figs. 2 and 5). Since, as $i \rightarrow \infty$, $h_d(U_i)$ and $h_d(R_i)$ approach $\pm\infty$ and $h_d(W_i)$ and $h_d(S_i)$ approach $\mp\infty$, each triangle includes definitely the first one when $y_{bi} > y_{r2}$. This limit y_{r2} is the bigger among y_{b1} and the largest root of the derivatives with respect to y_{bi} of $h_d(U_i)$ and $h_d(R_i)$, given by (2.24) and (2.25). Therefore, it is not necessary to examine the triangles for $y_{bi} > y_{r2}$.

A second-order plant, whose transfer function is without zeros, is considered as example and the stability regions of the plant and the controller parameters are depicted respectively in Figs. 4 and 5. In Fig. 4 this region consists of the following:

- Z_1 : $U(n, n) > 0$; $\Phi_1 > 0$ and $\Phi_2 > 0$
- Z_2 : $U(n, n) > 0$; $\Phi_1 < 0$ and $\Phi_2 > 0$
- Z_3 : $U(n, n) < 0$; $\Phi_2 < 0$

where $\Phi_1 = 1 + t_1 + t_2$ and $\Phi_2 = 1 + 2t_1 + 2t_2 + 2t_1t_2$ according to (2.16) and (2.17). The required number N_e of the intersections between $E(y)$ and $\tan(y/2)$, equal to $4r$ as per (3.2), coincide with the actual number only in this zone.

Since $P(y) = 1 - t_1t_2y^2$ $Q(y) = (t_1 + t_2)y$, considering the point $(t_1 = 0.6; t_2 = 0.8)$ lying in Z_1 of Fig. 4, one obtains $y_p = 1.778$ from (2.19) for the first root $y = y_p$ of $dG_1(0)/dy$ and $h_p = G_1(y_p) = 2.330$ from (2.7). Since $-1 < h < h_p$ must hold, let us assume h equal to 0.5; for the first two roots higher than zero of $G(y)$ one obtains $y_{a1} = 0.863$ and $y_{b1} = 2.498$ from (2.5) and hence $h_{ea1} = F_1(y_{a1}) = 1.099$ and $h_{eb1} = F_1(y_{b1}) = -9.985$ from (2.6). From (2.23) and (2.24) it follows $h_i(V_1) = 2.600$, $h_d(V_1) = 2.016$, $h_d(U_1) = 1.600$ and $h_d(W_1) = -1.476$. Similarly, for the third and fourth roots of $G(y)$ one obtains $y_{a2} = 5.285$ and $y_{b2} = 8.191$, $h_{ea2} = 76.290$, $h_{eb2} = -272.288$, $h_d(U_2) = 4.058$ and $h_d(W_2) = -2.732$; moreover $h_d(R_2) = 4.097$ and $h_d(S_2) = -2.638$ from (2.25) and (2.26).

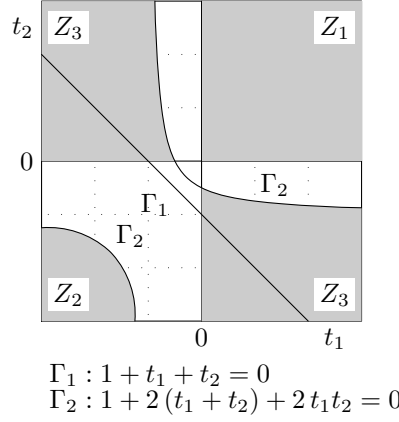


FIGURE 4. Stability zone of the process parameters for $n = 2$ and $P_n(s) = 1$

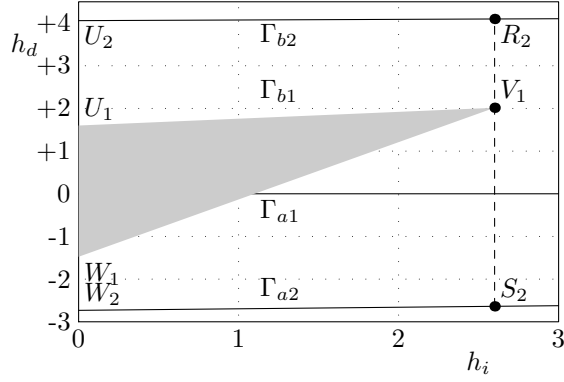


FIGURE 5. Stability zone of the controller parameters h_i and h_d for $n = 2$, $P_n(s) = 1$, $t_1 = 0.6$, $t_2 = 0.8$, $h = 0.5$

4. PROCESS TRANSFER FUNCTION WITH ZEROS

The function $H(\sigma)$, given by (2.3), can be rewritten as

$$(4.1) \quad H(\sigma) = \sigma^{n+1-m} e^{\sigma} \frac{U(n, n)}{V(m, m)} \frac{1 + H_n}{1 + H_d},$$

where

$$H_n = \sum_{i=1}^{i=n} \frac{U(n, n-i)}{U(n, n)\sigma^i} + e^{-\sigma} (h_i \sigma^{-2} + h \sigma^{-1} + h_d) \sum_{i=n-1-m}^{i=n-1} \frac{V(m, n-1-i)}{U(n, n)\sigma^i}$$

$$H_d = \sum_{i=1}^{i=m} \frac{V(m, m-i)}{V(m, m)\sigma^i}$$

$m < n - 1$ or $m = n - 1$ and $|h_d V(m, m)| < |U(n, n)|$, as explained in Section 2. Since the denominator of $H(\sigma)$ is the numerator of the plant transfer function, the poles of $H(\sigma)$ are the zeros of the plant transfer function ($\sigma_i = -1/z_i$). In this case

the proposed procedure will be in accordance with Theorem 4.1, a generalization of the theorems applied in Section 3 and of Theorem 13.5 of [9]; it will be here enunciated and proved by use of the Principle of the Argument.

Theorem 4.1. *Consider a function $H(\sigma)$ of the form in (4.1), set $H(jy) = F(y) + jG(y)$ and let ϵ be an appropriate constant such that the coefficient of y^{n+1-m} in the numerator of $G(y)$ does not vanish at $y = \epsilon$. Assume further that the m_p poles with positive real part of $H(\sigma)$ lie all in the rectangle R , described by the inequalities $-2r\pi + \epsilon \leq y \leq 2r\pi + \epsilon$, $x > 0$ ($\sigma = x + jy$). Suppose finally that the function $H(\sigma)$ does not assume the value of zero on the imaginary axis, that is, $H(jy) \neq 0$.*

All the zeros of $H(\sigma)$ lie to the left of the imaginary axis if and only if:

- (a) *The vector $w = H(jy)$ for real y ranging from $-\infty$ to $+\infty$ continually revolves in the positive direction at a positive velocity, that is, the inequality $G'(y_0)F(y_0) < 0$ is satisfied for each root $y = y_0$ of $G(y)$.*
- (b) *The number N_r of the roots of $G(y)$ in the interval $-2r\pi + \epsilon \leq y \leq 2r\pi + \epsilon$ for sufficiently large r is*

$$(4.2) \quad N_r = 4r + n + 1 - m + 2m_p.$$

Proof. Denote by C_1, C_2, C_3 and C_4 the corners of the rectangle R whose coordinates in the σ -plane are respectively $\sigma_1 = 0 + j(-2r\pi + \epsilon)$, $\sigma_2 = +\infty + j(-2r\pi + \epsilon)$, $\sigma_3 = +\infty + j(+2r\pi + \epsilon)$ and $\sigma_4 = 0 + j(+2r\pi + \epsilon)$; the arguments $\theta(\sigma_1)$ and $\theta(\sigma_4)$ of $H(\sigma)$ are given by $\theta(\sigma_1) = -2r\pi + \epsilon - 0.5(n + 1 - m)\pi + \eta + \delta_1$ and $\theta(\sigma_4) = +2r\pi + \epsilon + 0.5(n + 1 - m)\pi + \eta + \delta_4$, where $\eta = 0$ if $V(m, m)/U(n, n) > 0$ or $\eta = \pi$ if $V(m, m)/U(n, n) < 0$, and $\delta_1 \rightarrow 0$ and $\delta_4 \rightarrow 0$ simultaneously with $1/r$. Denote by N_z the number of the zeros of $H(\sigma)$ lying in the rectangle R , by V_a the variation of the argument of $H(\sigma)$ in the counterclockwise direction as σ moves around the contour of R from C_1 to C_4 through C_2 and C_3 , and by V_b as σ moves directly from C_4 to C_1 . Using the Principle of the Argument yields

$$(4.3) \quad V_a + V_b = (N_z - m_p)2\pi.$$

Since $N_z = 0$ for stability and, for $r \rightarrow \infty$, $V_b = -N_r\pi$ as per condition (a) and $V_a = \theta_4 - \theta_1 = +4r\pi + (n + 1 - m)\pi$, (4.2) follows from (4.3) and the condition (b) is satisfied. \square

The procedure detailed in Section 3 for process transfer functions without zeros is fully applicable to process transfer functions with zeros; it is only necessary to replace (3.1) with (4.2) and to consider $C(y)$ and $D(y)$ as functions given by (2.11) and (2.12) instead of $C(y) = 1$ and $D(y) = 0$. Moreover, for $m = n - 1$, the rectangle, according to (3.3) for $y_0 = +\infty$ and provided with horizontal sides symmetric with respect to the axis h_i at a distance given by (2.27), must be considered in the (h_i, h_d) -plane.

5. CONCLUSIONS

In this note, both stable and unstable delay-free arbitrary-order plants, provided with one time delay and PID controller, have been examined and the related stability regions in process and controller parameter spaces have been determined by use of the Pontryagin's studies. The proposed procedure, consisting of a finite number of steps, yields explicit expressions of the boundaries of the stability zone for the controller parameters. These results can be implemented in tuning charts, which become a complete tool for the design and the maintenance of control systems.

APPENDIX A. STURM THEOREM

The Sturm Theorem states that the number of real roots of an algebraic equation with real coefficients whose real roots are simple over an interval, the endpoints of which are not roots, is equal to the difference between the numbers of sign changes of the Sturm chains formed for the interval ends.

Given a function $f(x) = f_0(x)$ of degree d , assume $f_1(x) = df_0(x)/dx$ and define the Sturm functions by

$$f_i(x) = -f_{i-2}(x) + f_{i-1}(x) \left[\frac{f_{i-2}(x)}{f_{i-1}(x)} \right],$$

where $[f_{i-2}(x)/f_{i-1}(x)]$ is a polynomial quotient. These functions can be written as

$$f_i(x) = \sum_{j=d-\text{int}((i+1)/2)}^{j=d-\text{int}((i+1)/2)} \psi_{i,j} x^j \quad 0 \leq i \leq 2d-1,$$

where $\psi_{i,j}$ depends on the coefficients of x in $f(x)$.

In our case $x = y^2$ and $f_0(x) = C^2(\sqrt{x}) + D^2(\sqrt{x}) + A(\sqrt{x})C(\sqrt{x}) + B(\sqrt{x})D(\sqrt{x})$ hold; since the roots must be positive, the required interval is from $x = 0$ to $x = +\infty$ and, therefore, the signs of each Sturm function at these ends are the signs of $\psi_{i,j}$ evaluated respectively for $j = 0$ and $j = d - \text{int}((i+1)/2)$.

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